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Engineering Notes

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Reducing the Numerical Viscosity in Nonstructured Three-Dimensional Finite Volume Computations

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Introduction

WHEN solving numerically the fluid mechanic equations using finite volume techniques, the necessity of computing convective fluxes arises. Traditional approaches for computing such fluxes give good results if the variables undergo smooth variations, however, they have serious difficulties if the solution contains discontinuities. In these cases, the numerical schemes that use second- or higher-order approximations develop convergence problems and the solution has oscillations next to discontinuities. On the other hand, the schemes that use first-order approximations generate solutions without oscillations, but the discontinuities may poorly be resolved. To deal with this problem, flux limiter functions were built as linear combinations of first- and second-order approximations [1,2]. In this lineal combination, if the first-order approach has more weight than that of the second order, the scheme becomes diffusive and, reciprocally, if the second-order approach has more weight, the scheme becomes compressive.

In robust schemes, the number of limiter functions is equal to the number of equations and a spectral decomposition is used [3]. For the three-dimensional Euler equations system, the spectral decomposition leads to the appearance of three lineally degenerate families of waves [4]. Discontinuities associated with these waves are very difficult to resolve except for schemes that use higher compressive limiters, however, schemes with these limiters are not very robust in solving discontinuities associated with the nonlineal wave families [2].

In this work, a scheme is described which has the capacity to satisfactorily solve discontinuities associated with lineally degenerate wave families without losing robustness. It was implemented in a computational code that, using a nonstructured finite volume technique, solves the three-dimensional Euler equations. Results obtained in some applications are presented.

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Description of the Scheme

The three-dimensional Euler equations can be written as

$$\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F} = 0 \tag{1}$$

where $\boldsymbol{U}$ is the vector of conservative variables, and $\boldsymbol{F}$ is the three-dimensional vectorial flow.

The temporal change of the conservative variables can be written as

$$U^{n+1} = U^n - \frac{\Delta t}{\mathrm{Vol}} \cdot \sum_{l=1}^{l\mathrm{faces}} F_l^* \bullet n_l \cdot A_l = 0 \tag{2}$$

Equation (2) allows the use of a locally aligned system of coordinates whose unit vector $\boldsymbol{i}$ coincides with the normal to the face "l" of the cell, and the unit vectors $\boldsymbol{j}$ and $\boldsymbol{k}$ are tangential directions. Because the local Riemann problem is solved with rotated data, the eigensystem is calculated in the locally aligned coordinate frame. To achieve second-order accuracy and to retain total variation diminishing (TVD) properties, the numerical flux can then be written as

$$f_{i+1/2}^* = \frac{f_i + f_{i+1}}{2} + \frac{1}{2} \cdot \sum_m \Phi_{i+1/2}^m \cdot \overline{r_{i+1/2}^m} \tag{3}$$

where f_i and f_{i+1} are the physical fluxes normal to the face in each cells, $\overline{r_{i+1/2}^m}$ is the mth right eigenvector, and $\Phi_{i+1/2}^m$ is, in the TVD scheme proposed by Harten [5,6] and modified by Yee et al. [7], defined as

$$\Phi_{i+1/2}^m = g_i^m + g_{i+1}^m - |\lambda_{i+1/2}^m + \gamma_{i+1/2}^m| \cdot \alpha_{i+1/2}^m \tag{4}$$

being

$$g_i^m = \frac{\mathrm{sgn}\,(\lambda_{i+1/2}^m)}{2} \cdot \max\Big[0, \min\Big(|\lambda_{i+1/2}^m \cdot \alpha_{i+1/2}^m|, \mathrm{sgn}\,\big(\lambda_{i+1/2}^m\big) \cdot |\lambda_{i-1/2}^m| \cdot \alpha_{i-1/2}^m\Big)\Big] \tag{5}$$

$$\gamma_{i+1/2}^m = \frac{g_{i+1}^m - g_i^m}{\alpha_{i+1/2}^m} \quad \text{if } \alpha_{i+1/2}^m \neq 0 \quad \text{or} \quad \gamma_{i+1/2}^m = 0 \quad \text{if } \alpha_{i+1/2}^m = 0 \tag{6}$$

where $\alpha_{i+1/2}^m$ is the jump of the conserved variables across the interfaces between cells i and $i+1$, and $\lambda_{i-1/2}^m$ is the mth eigenvalue of the Jacobian matrix. The limiter defined in Eq. (5) is known as "minmod" [1–4].

In any numerical solution of the three-dimensional Euler equations, five wave families enter into consideration. If these five wave families are enumerated in correspondence with their speed, one being the slowest and five the fastest, it can be demonstrated that for waves of the families 2–4, the characteristic velocities at both sides of a discontinuity are the same and equal to the discontinuity velocity [2,3]. This property makes it very difficult to numerically resolve these waves accurately, unless it is done diffusely.

Because in this work, the possibility of implementing different limiter functions for different wave families is explored, to numerically resolve discontinuities associated with the families 2–4, the limiter "superbee" will be used. It can be introduced in the numerical fluxes calculations replacing Eq. (5) by the following expression

$$g_i^m = \max[0, \min(2\cdot r, 1), \min(r, 2)]\cdot \frac{1}{2}\cdot |\lambda_{i-1/2}^m|\cdot \alpha_{i-1/2}^m \quad \text{if } \alpha_{i+1/2}^m\cdot \alpha_{i-1/2}^m \ge 0$$
$$g_i^m = 0 \quad \text{if } \alpha_{i+1/2}^m\cdot \alpha_{i-1/2}^m < 0 \tag{7}$$

being

$$r = |\lambda_{i+1/2}^m|\cdot \alpha_{i+1/2}^m / |\lambda_{i-1/2}^m|\cdot \alpha_{i-1/2}^m$$

Notice that different limiters are only applied on those cells where strong discontinuities in central waves are detected, otherwise, the conventional Harten–Yee scheme is used.

The comparison among the intensity of the waves is performed measuring the density jumps in waves 1, 2, and 5, and the momentum jumps in waves 3 and 4. Such intensities can be expressed as follows:

$$I_1 = \frac{|\alpha_{i+1/2}^1|}{\rho_{\text{ref}}}, \qquad I_2 = \frac{|\alpha_{i+1/2}^2|}{\rho_{\text{ref}}}, \qquad I_3 = \frac{|\alpha_{i+1/2}^3|}{\rho_{\text{ref}} u_{\text{ref}}}$$
$$I_4 = \frac{|\alpha_{i+1/2}^4|}{\rho_{\text{ref}} u_{\text{ref}}}, \qquad I_5 = \frac{|\alpha_{i+1/2}^5|}{\rho_{\text{ref}}} \tag{8}$$

where the selected references values for the density and velocity are $\rho_{\text{ref}} = 0.5(\rho_i + \rho_{i+1})$ and $u_{\text{ref}} = 0.5(c_i + c_{i+1})$.

If the maximum of I_1, I_5 is higher than the maximum of I_2, I_3, I_4, the conventional Harten–Yee TVD scheme is used; otherwise, the values of g_i^2, g_i^3, g_i^4 are calculated with the limiter function superbee and g_i^1, g_i^5 with the limiter function minmod.

For the evaluation of g_i^m and g_{i+1}^m in Eq. (4), it is necessary to calculate the spectral decompositions of the conservative variables increments at the interfaces $i-1/2$, $i+1/2$, and $i+3/2$. In the context of three-dimensional unstructured meshes of tetrahedrons, the identification of the cells i and $i+1$ is intuitive (they are two cells that share a face), but the determination of points $i-1$ and $i+2$ is not direct. If two tetrahedrons that share a face are analyzed, the nodes not belonging to the common face can be used as representative points for $i-1$ and $i+2$. This idea has been implemented and the nodal values are calculated as a pondered average of conservative variables between all cells that are in contact with nodes $i-1$ and $i+2$. Such pondered average is given by

$$U_{\text{node } k} = \left(\sum_{i=1}^{n} U_{\text{cell } i}/d_{\text{cg}_{\text{cell } i}-\text{node } k}\right) \Big/ \left(\sum_{i=1}^{n} 1/d_{\text{cg}_{\text{cell } i}-\text{node } k}\right)$$

where $d_{\text{cg}_{\text{cell } i}-\text{node } k}$ is the distance that separates the gravity center of cell i from the node k, and n is the cell number in contact with node k.

Test Cases

To verify the accuracy and robustness of the adaptive scheme, two test cases are simulated. The first one is the flow inside of a shock tube. This example is used to explore the capacity of the scheme which has been described, to model contact discontinuities, and nonlineal waves simultaneously. The second test is the simulation of a slip layer between two flows with different velocities and densities, and equal pressures. This test was chosen to study the capacity of the scheme to solve flows in which the discontinuity is in the velocity tangent to a given interface (discontinuity in waves of the families 3 and 4).

Flow Inside a Shock Tube

The shock tube has a rectangular section of 4×4 cm and a length of 2 m; the gas is air. The driver section is filled with atmospheric initial conditions. The pressure in the driven section is 0.1 atm and possesses the same temperature as the driver. The analytical solution is given by a three wave system: a shock wave that advances toward the right at 543.4 m/s, a contact discontinuity that also moves toward the right at 277.6 m/s, and an expansion fan whose head wave travels to the left at 4.9 m/s and the tail wave also traveling leftward at 338.1 m/s [3]. The computational mesh is obtained by means of a "generating cell" built up by six tetrahedrons that form a cube of 1 cm side length. The cube is repeated to complete an arrangement of $4 \times 4 \times 200$.

Figures 1 and 2 show the density as a function of (x/t). Figure 1 compares the results obtained applying the traditional Harten–Yee TVD scheme with those calculated using the method here proposed. It can be appreciated that the simulation of the nonlineal waves is the same for both numerical schemes. However, the capture of the contact discontinuity has been improved notably. Figure 2 compares the results obtained applying the scheme proposed in this work with those obtained using superbee limiter functions in all waves. It can be appreciated that the results obtained using the flux limiters superbee show oscillations, which are not present using the proposed scheme. It is also worth noticing the accuracy achieved in computing the speed of the waves.

Slip Surface

The second test case is an air layer with a density of 1.225 kg/m^3, a velocity of 1690.3 m/s, and a pressure of 100,000 Pa, which flows over another layer with a density of 3 kg/m^3, a velocity of 648.1 m/s, and a pressure also of 100,000 Pa. The analytical solution predicts the slip of a flow on another without interferences.

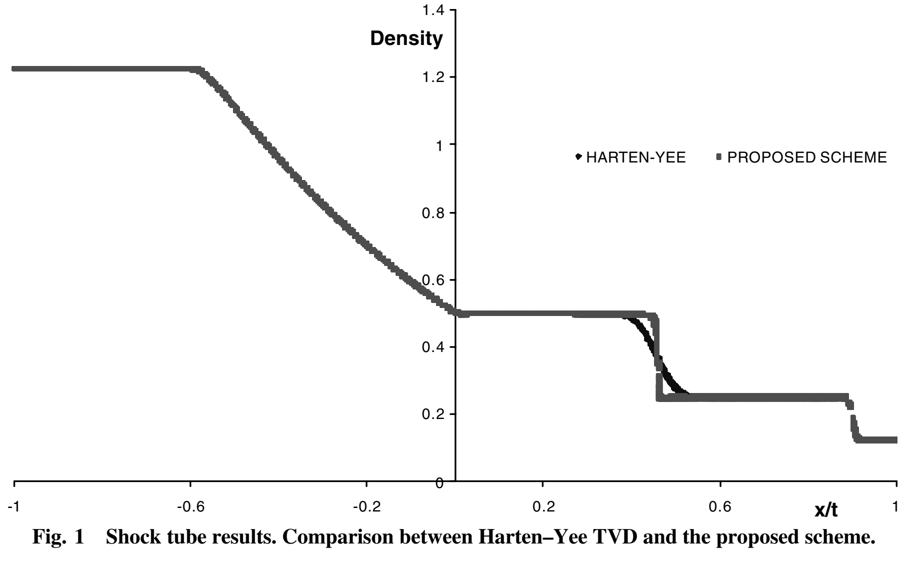

Fig. 1 Shock tube results. Comparison between Harten–Yee TVD and the proposed scheme.

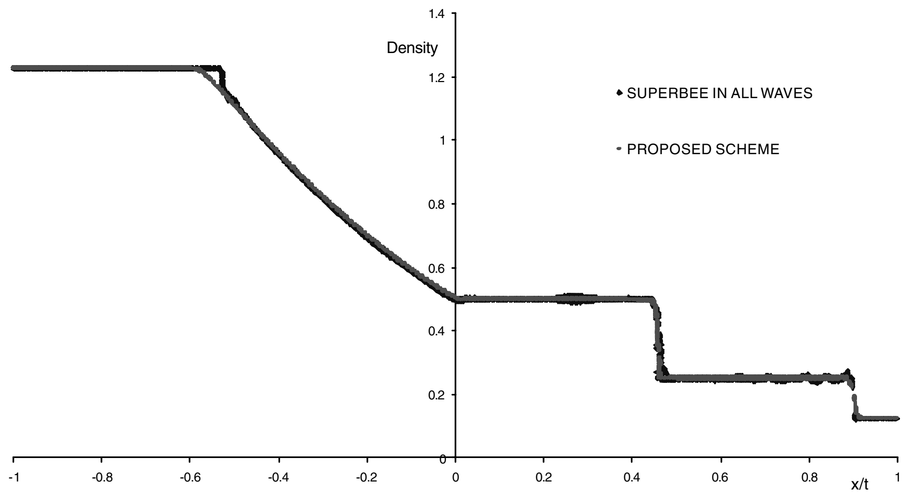

Fig. 2 Shock tube results. Comparison between scheme using superbee limiter functions in all waves and the proposed scheme.

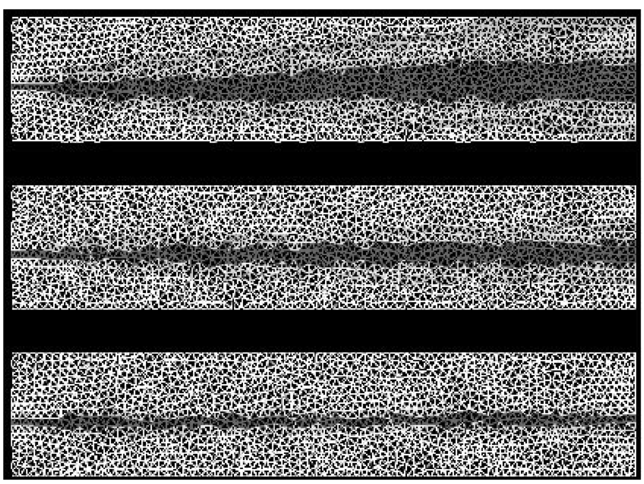

Fig. 3 Error distribution in slip surface results. Upper: Harten–Yee; middle: new scheme; lower: superbee in all waves.

However, due to the numerical viscosity, the computed solution produces an apparent mixture zone that gets wider downstream of the end of the splitter plane. The spreading of such unphysical mixing region quantifies the lack of accuracy of the numerical method.

The used mesh has 57,790 tetrahedrons and 11,503 nodes. The control volume where the flow develops is 5 m long, 1 m high, and 1 m wide. The nonstructured mesh does not have any bias plane that may influence the location of the slip discontinuity.

Figure 3 presents the absolute value of the percentage of error incurred in the velocity predicted at a lateral plane of the computational domain. The results using the traditional Harten–Yee TVD scheme are shown in the upper part, those obtained with the scheme that is proposed in the middle, and shown in the lower part are the results calculated using superbee limiter functions in all waves. The error is calculated by

$$e_{\%} = 100|u_n - u_{t(z)}|/\bar{u}$$

u_n being the velocity calculated numerically, $u_{t(z)} = 1690.3$ m/s the velocity if the z coordinate of the center of gravity (c.g.) of the cell is higher than 0.5 m, or $u_{t(z)} = 648.1$ m/s if the cell c.g. z coordinate is smaller than 0.5 m, and $\bar{u} = 0.5(1690.3 + 648.1\ \mathrm{m/s}) = 1169.2$ m/s.

It has been observed that the angle formed by the straight lines that limit the area with errors higher than 2% is 9.9 deg for the Harten–Yee scheme, whereas the angle reduces to 4.8 deg for the scheme that is here proposed. However, using flux limiters superbee in all waves, the angle is approximately 2.2 deg. Consequently, the proposed scheme is notably less diffusive than the conventional Harten–Yee method, although more diffusive than the scheme using superbee limiters.

Conclusions

The results obtained for the shock tube have shown that the scheme which has been proposed diminishes the numerical viscosity with regards to the conventional Harten–Yee TVD scheme and it does not introduce oscillations. Furthermore, the ability in capturing the shock wave has not been affected.

For the slip surface test, where only discontinuities in waves of families 3 and 4 are involved, the results have shown that the proposed scheme works more efficiently than the Harten–Yee conventional one. However, the scheme using superbee limiter functions in all waves produces less diffusion. The superior behavior of the flux limiters superbee is mainly due to the fact that, in this test case, there are not shock waves. When shocks are present, the use of the superbee functions introduce spurious oscillations and instabilities.

Acknowledgments

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